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STABILITY OF TWO GENERALIZED 3-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATIONS

Sun-Sook Jin* and Yang-Hi Lee**

Abstract. In this paper, we investigate the stability of two functional equations

$$f(ax+by+cz) - abf(x+y) - bcf(y+z) - acf(x+z) + bcf(y) \\ - a(a-b-c)f(x) - b(b-a)f(-y) - c(c-a-b)f(z) = 0,$$

$$f(ax+by+cz) + abf(x-y) + bcf(y-z) + acf(x-z) \\ - a(a+b+c)f(x) - b(a+b+c)f(y) - c(a+b+c)f(z) = 0$$

by applying the direct method in the sense of Hyers and Ulam.

1. Introduction

In 1941, Hyers [3] gave an affirmative answer to Ulam's stability problem of the group homomorphisms[9] for additive mappings between Banach spaces. Subsequently many mathematicians dealt with this problem (cf. [1, 2, 8]).

A solution of the functional equation

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{1.1}$$

is called a quadratic mapping. Now we consider the following functional equations

$$f(ax+by+cz) - abf(x+y) - bcf(y+z) - acf(x+z) + bcf(y) \\ - a(a-b-c)f(x) - b(b-a)f(-y) - c(c-a-b)f(z) = 0 \tag{1.2}$$

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Correspondence should be addressed to Yang-Hi Lee, lyhmzi@pro.gjue.ac.kr.

and

$$\begin{aligned} &f(ax+by+cz)+abf(x-y)+bcf(y-z)+acf(x-z)\\ (1.3)\qquad &-a(a+b+c)f(x)-b(a+b+c)f(y)-c(a+b+c)f(z)=0 \end{aligned}$$

for nonzero rational numbers a, b, c. The mapping $f(x) = dx^2$ is a solution of these functional equations, where d is a real constant. The authors [4, 5] investigated the stability of the equation (1.2) for the cases $a = b = c$ and $a = b = c = \frac{1}{3}$ and they [6] also investigated the stability of the equation (1.3) for the case $c = 0$ (see also [7]). Let a, b, c be nonzero rational numbers.

In this paper, we will show that every solution of functional equation (1.2) is a quadratic mapping and every quadratic mapping is a solution of functional equation (1.2) for the case $a \neq b$. In this paper, we will show that every solution of functional equation (1.3) is a quadratic mapping and every quadratic mapping is a solution of functional equation (1.3) for the case $a \neq -b$. Also we will prove the stability of the functional equations (1.2) by using the Hyers' method presented in [3]. Namely, starting from the given mapping f that approximately satisfies the functional equation (1.2), a solution F of the functional equation (1.2) is explicitly constructed by using the formula

$$F(x) := \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}} \text{ or } F(x) := \lim_{n\to\infty} a^{2n} f\Big(\frac{x}{a^n}\Big),$$

which approximates the mapping f.

2. Stability of the functional equation (1.2)

Throughout this paper, let V and W be (real or complex) vector spaces, let X be a (real or complex) normed space, let Y be a Banach space, and let a, b, c be nonzero rational numbers.

For a given mapping $f : V \to W$ and, we use the following abbreviations

$$\begin{aligned} &Qf(x,y) := f(x+y)+f(x-y)-2f(x)-2f(y),\\ &D_{a,b}f(x,y) :=\\ &\qquad f(ax+by)-abf(x+y)-(a^2-ab)f(x)-(b^2-ab)f(-y),\\ &D_{a,b,c}f(x,y,z) :=\\ &\qquad f(ax+by+cz)-abf(x+y)-bcf(y+z)-acf(x+z)\\ &\qquad -a(a-b-c)f(x)-b(b-a)f(-y)+bcf(y)-c(c-a-b)f(z), \end{aligned}$$

$$
\begin{aligned}
E_{a,b}f(x,y) :=& \\
&f(ax+by)+abf(x-y)-(a^2+ab)f(x)-(b^2+ab)f(y),\\
E_{a,b,c}f(x,y,z) :=& \\
&f(ax+by+cz)+abf(x-y)+bcf(y-z)+acf(x-z)\\
&-a(a+b+c)f(x)-b(a+b+c)f(y)-c(a+b+c)f(z)
\end{aligned}
$$

for all $x, y, z \in V$. As we stated in the previous section, a solution of $Qf = 0$ is called a quadratic mapping. Now we will show that f is a quadratic mapping if f is a solution of the functional equation $D_{a,b,c}f(x,y,z) = 0$ for all $x, y, z \in V$.

Lemma 2.1. [6] *Let a and b be fixed nonzero rational numbers with $a+b \neq 0$. A mapping $f : V \to W$ is a solution of the functional equation*

$$E_{a,b}f(x,y) = 0$$

(with $f(0) = 0$ when $a^2+ab+b^2 = 1$) if and only if f is a quadratic mapping.

From the above lemma, we easily obtain the following lemma.

Lemma 2.2. *Let a and b be fixed nonzero rational numbers with $a \neq b$. A mapping $f : V \to W$ is a solution of the functional equation*

$$D_{a,b}f(x,y) = 0$$

(with $f(0) = 0$ when $a^2-ab+b^2 = 1$) if and only if f is a quadratic mapping.

Since the authors [4] showed the stability of the equation (1.2) for the cases $a = b = c$, we need to prove the stability of the equation (1.2) for the cases $a \neq b$ or $b \neq c$. We can assume that $a \neq b$ without loss of generality from the symmetry of a and c.

Lemma 2.3. *Let a, b and c be nonzero rational numbers such that $a \neq b$. A mapping $f : V \to W$ satisfies the functional equation $D_{a,b,c}f(x,y,z) = 0$(with $f(0) = 0$ when $a^2+b^2+c^2-ab-bc-ac = 1$) if and only if f is a quadratic mapping.*

Proof. If $a^2+b^2+c^2-ab-bc-ac \neq 1$, then $(1-a^2-b^2-c^2+ab+bc+ac)f(0) = D_{a,b,c}f(0,0,0) = 0$ which means that $f(0) = 0$. If $f : V \to W$ is a solution of the functional equation $D_{a,b,c}f(x,y,z) = 0$, then the equality $D_{a,b}f(x,y) = D_{a,b,c}f(x,y,0) = 0$ implies that f is a quadratic mapping by Lemma 2.2.

Conversely, let $f : V \to W$ be a quadratic mapping. Then $f(0) = 0$, $f(x) = f(-x)$, $f(ax) = a^2 f(x)$, $f(bx) = b^2 f(x)$, $f(cx) = c^2 f(x)$, $D_{a,a} f(x,y) = 0$, and $D_{b,b} f(x,y) = 0$. By Lemma 2.2, we know that f satisfies the functional equations $D_{a,b} f(x,y) = 0$, $D_{a,c} f(x,y) = 0$, $D_{b,c} f(x,y) = 0$, where a, b, c are arbitrary different rational constants. So we obtain the equality

$$\begin{aligned} &D_{a,b,c} f(x,y,z) \\ &\quad = Qf\left(ax + \frac{cz}{2}, by + \frac{cz}{2}\right) - Qf\left(ax + \frac{cz}{2}, \frac{cz}{2}\right) - Qf\left(by + \frac{cz}{2}, \frac{cz}{2}\right) \\ &\qquad - Qf(ax, by) + D_{a,b} f(x,y) + D_{a,c} f(x,z) + D_{b,c} f(y,z) \\ &\qquad + f(ax) + f(by) + 4f(\frac{cz}{2}) - a^2 f(x) - b^2 f(y) - c^2 f(z) = 0 \end{aligned}$$

for all $x, y, z \in V$. □

THEOREM 2.4. *Let a, b and c be nonzero rational numbers with $a \neq b$ and let $\varphi : V^3 \to [0, \infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x, a^i y, a^i z)}{a^{2i}} < \infty, \tag{2.1}$$

$$\sum_{i=0}^{\infty} a^{2i} \varphi\left(\frac{x}{a^i}, \frac{y}{a^i}, \frac{z}{a^i}\right) < \infty \tag{2.2}$$

for all $x, y, z \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and

$$\|D_{a,b,c} f(x,y,z)\| \le \varphi(x,y,z) \tag{2.3}$$

for all $x, y, z \in V$, then there exists a unique quadratic mapping $F : V \to Y$ such that

(2.4)
$$\|f(x) - F(x)\| \le \begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x, 0, 0)}{a^{2i+2}} & \textit{if } \varphi \textit{ satisfies } (2.1), \\ \sum_{i=0}^{\infty} a^{2i} \varphi\left(\frac{x}{a^{i+1}}, 0, 0\right) & \textit{if } \varphi \textit{ satisfies } (2.2) \end{cases}$$

for all $x \in V$.

Proof. We will prove the theorem in two cases, either φ satisfies (2.1) or φ satisfies (2.2).

Case 1. Let φ satisfy (2.1). It follows from (2.3) that

$$\begin{aligned}\left\|\frac{f(a^n x)}{a^{2n}} - \frac{f(a^{n+m}x)}{a^{2n+2m}}\right\| &= \sum_{i=n}^{n+m-1}\left\|\frac{f(a^i x)}{a^{2i}} - \frac{f(a^{i+1}x)}{a^{2i+2}}\right\| \\ &\le \sum_{i=n}^{n+m-1}\frac{\|-D_{a,b,c}f(a^i x,0,0)\|}{a^{2i+2}} \\ &\le \sum_{i=n}^{n+m-1}\frac{\varphi(a^i x,0,0)}{a^{2i+2}} \qquad (2.5)\end{aligned}$$

for all $x \in V$, $n \in \mathbb{N}\cup\{0\}$ and $m \in \mathbb{N}$. So, it is easy to show that the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete and $f(0) = 0$, the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty}\frac{f(a^n x)}{a^{2n}}$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (2.5), we obtain the first inequality in (2.4). From the definition of F and (2.3), we get

$$\begin{aligned}\|D_{a,b,c}F(x,y,z)\| &= \lim_{n\to\infty}\left\|\frac{D_{a,b,c}f\big(a^n x, a^n y, a^n z\big)}{a^{2n}}\right\| \\ &\le \lim_{n\to\infty}\frac{\varphi\big(a^n x, a^n y, a^n z\big)}{a^{2n}} = 0,\end{aligned}$$

i.e., $D_{a,b,c}F(x,y,z) = 0$ for all $x,y,z \in V$. By Lemma 2.3, f is a quadratic mapping. To prove the uniqueness, we assume now that there is another quadratic mapping $F' : V \to W$ which satisfies the first inequality in (2.4). Notice that $F'(x) = \frac{F'(a^n x)}{a^{2n}}$ for all $x \in V$. Using (2.1) and (2.4), we obtain

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f(a^n x)}{a^{2n}} - F'(x)\right\| &= \lim_{n\to\infty}\left\|\frac{f(a^n x)}{a^{2n}} - \frac{F'(a^n x)}{a^{2n}}\right\| \\ &\le \lim_{n\to\infty}\sum_{i=0}^{\infty}\frac{\varphi(a^{i+n}x,0,0)}{a^{2n+2i+2}} \\ &\le \lim_{n\to\infty}\sum_{i=n}^{\infty}\frac{\varphi(a^i x,0,0)}{a^{2i+2}} \\ &= 0\end{aligned}$$

for all $x \in V$, i.e, $F'(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}} = F(x)$ for all $x \in V$.

Case 2. Let φ satisfy (2.2). It follows from (2.3) that

$$
\begin{aligned}
&\left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2n+2m} f\left(\frac{x}{a^{n+m}}\right)\right\| \\
&\qquad = \sum_{i=n}^{n+m-1} \left\| a^{2i} f\left(\frac{x}{a^i}\right) - a^{2i+2} f\left(\frac{x}{a^{i+1}}\right)\right\| \\
&\qquad \le \sum_{i=n}^{n+m-1} a^{2i} \left\| D_{a,b,c} f\left(\frac{x}{a^{i+1}}, 0, 0\right)\right\| \\
&\qquad \le \sum_{i=n}^{n+m-1} a^{2i} \varphi\left(\frac{x}{a^{i+1}}, 0, 0\right) \qquad (2.6)
\end{aligned}
$$

for all $x \in V$, $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. So, it is easy to show that the sequence $\{a^{2n} f(\frac{x}{a^n})\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete and $f(0) = 0$, the sequence $\{a^{2n} f(\frac{x}{a^n})\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty} a^{2n} f\left(\frac{x}{a^n}\right)$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (2.6), we obtain the second inequality in (2.4). From the definition of F and (2.3), we get

$$
\begin{aligned}
\|D_{a,b,c} F(x, y, z)\| &= \lim_{n\to\infty} \left\| a^{2n} D_{a,b,c} f\left(\frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n}\right)\right\| \\
&\le \lim_{n\to\infty} a^{2n} \varphi\left(\frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n}\right) \\
&= 0
\end{aligned}
$$

for all $x, y, z \in V$ i.e., $D_{a,b,c} F(x, y, z) = 0$ for all $x, y, z \in V$. By Lemma 2.3, f is a quadratic mapping. To prove the uniqueness, we assume now that there is another mapping $F' : V \to W$ which satisfies the second inequality in (2.4). Notice that $F'(x) = a^{2n} F'(\frac{x}{a^n})$ for all $x \in V$. Using Lemma 2.1, (2.2), and (2.4), we get

$$
\begin{aligned}
&\lim_{n\to\infty} \left\| a^{2n} f\left(\frac{x}{a^n}\right) - F'(x)\right\| \\
&\qquad = \lim_{n\to\infty} \left\| a^{2n} f\left(\frac{x}{a^n}\right) - a^{2n} F'\left(\frac{x}{a^n}\right)\right\|
\end{aligned}
$$

$$\leq \lim_{n\to\infty} \sum_{i=0}^{\infty} a^{2n+2i}\varphi\bigg(\frac{x}{a^{n+i}}, 0, 0\bigg)$$
$$\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} a^{2i}\varphi\bigg(\frac{x}{a^{i}}, 0, 0\bigg)$$
$$= 0$$

for all $x \in V$, i.e., $F'(x) = \lim_{n\to\infty} a^{2n} f\big(\frac{x}{a^n}\big) = F(x)$ for all $x \in V$. □

We easily obtain the following theorems by using the similar method used in Theorem 2.4.

Theorem 2.5. *Let a, b and c be nonzero rational numbers with $a \neq b$ and let $\varphi : V^3 \to [0, \infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(c^i x, c^i y, c^i z)}{c^{2i}} < \infty, \tag{2.7}$$

$$\sum_{i=0}^{\infty} c^{2i}\varphi\bigg(\frac{x}{c^i}, \frac{y}{c^i}, \frac{z}{c^i}\bigg) < \infty \tag{2.8}$$

for all $x, y, z \in V$. If a mapping $f : V \to Y$ satisfies (2.3) for all $x, y, z \in V$ with $f(0) = 0$, then there exists a unique quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \sum_{i=0}^{\infty} \frac{1}{c^{2i+2}}\varphi(0, 0, c^i x) & \text{if } \varphi \text{ satisfites (2.7)}, \\ \sum_{i=0}^{\infty} c^{2i}\varphi\Big(0, 0, \frac{x}{c^{i+1}}\Big) & \text{if } \varphi \text{ satisfites (2.8)} \end{cases}$$

for all $x \in V$.

Corollary 2.6. *Suppose that a, b, c are given as in Theorem 2.4 and p, θ are positive real constants with $p \neq 2$. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|D_{a,b,c}f(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \min\left\{\frac{\theta\|x\|^p}{|a^2 - |a|^p|}, \frac{\theta\|x\|^p}{|c^2 - |c|^p|}\right\} \tag{2.9}$$

for all $x \in X$.

Proof. First, consider the case $|a|, |c| > 1$. If we put $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$, then φ satisfies (2.1) and (2.7) when $0 < p < 2$ and φ satisfies (2.2) and (2.8) when $p > 2$. Therefore by Theorems 2.4 and 2.5, we obtain the desired inequality (2.9). For the other cases, we can get easily the inequality (2.9) by the similar method. □

3. Stability of the functional equation (1.3)

In this section, we will show that f is a quadratic mapping if f is a solution of the functional equation $E_{a,b,c}f(x,y,z) = 0$ for all $x, y, z \in V$.

LEMMA 3.1. *Let a, b and c be nonzero rational numbers. A mapping $f : V \to W$ satisfies the functional equation $E_{a,b,c}f(x,y,z) = 0$(with $f(0) = 0$ when $a^2 + b^2 + c^2 + ab + bc + ac = 1$) if and only if f is a quadratic mapping.*

Proof. If $a^2+b^2+c^2+ab+bc+ac \neq 1$, then $(1-a^2-b^2-c^2-ab-bc-ac)f(0) = E_{a,b,c}f(0,0,0) = 0$ which means that $f(0) = 0$. If $f : V \to W$ is a solution of the functional equation $E_{a,b,c}f(x,y,z) = 0$ with $a+b \neq 0$, then the equality $E_{a,b}f(x,y) = E_{a,b,c}f(x,y,0) = 0$ implies that f is a quadratic mapping by Lemma 2.1. If $f : V \to W$ is a solution of the functional equation $E_{a,b,c}f(x,y,z) = 0$ with $a+b = 0$, then the equality $ac(f(x) - f(-x)) = E_{a,b,c}f(0,-x,0) - E_{a,b,c}f(x,0,0) = 0$ implies that $f(x) = f(-x)$ for all $x \in V$. Since a, b, c are nonzero rational numbers and $a+b = 0$, we know that $a+c \neq 0$ or $b+c \neq 0$. Without of generality, assume that $a+c \neq 0$, then the equality $E_{a,c}f(x,y) = E_{a,b,c}f(x,0,y) = 0$ implies that f is a quadratic mapping by Lemma 2.1.

Conversely, let $f : V \to W$ be a quadratic mapping. Then $f(0) = 0$, $f(x) = f(-x)$, $f(ax) = a^2 f(x)$, $f(bx) = b^2 f(x)$, $f(cx) = c^2 f(x)$, $E_{a,-a}f(x,y) = 0$, $E_{b,-b}f(x,y) = 0$, and $E_{c,-c}f(x,y) = 0$. By Lemma 2.1, we know that f satisfies the functional equations $E_{a,b}f(x,y) = 0$, $E_{a,c}f(x,y) = 0$, $E_{b,c}f(x,y) = 0$, where a, b, c are arbitrary rational constants. So we obtain the equality

$$\begin{aligned}
E_{a,b,c}f(x,y,z) = {} & Qf\left(ax + \frac{cz}{2}, by + \frac{cz}{2}\right) - Qf\left(ax + \frac{cz}{2}, \frac{cz}{2}\right) \\
& - Qf\left(by + \frac{cz}{2}, \frac{cz}{2}\right) - Qf(ax, by) + E_{a,b}f(x,y) \\
& + E_{a,c}f(x,z) + E_{b,c}f(y,z) = 0
\end{aligned}$$

for all $x, y, z \in V$. □

We will prove the stability of the functional equation (1.3) for the case $a+b+c\neq 0$ in the following theorem.

Theorem 3.2. *Let a,b and c be nonzero rational numbers with $a+b+c\neq 0$ and let $\varphi : V^3 \to [0,\infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(k^i x, k^i y, k^i z)}{k^{2i}} < \infty, \tag{3.1}$$

$$\sum_{i=0}^{\infty} k^{2i}\varphi\Big(\frac{x}{k^i}, \frac{y}{k^i}, \frac{z}{k^i}\Big) < \infty \tag{3.2}$$

for all $x,y,z\in V$, where $k := a+b+c$. If a mapping $f : V \to Y$ satisfies $f(0)=0$ and

$$\|E_{a,b,c}f(x,y,z)\| \leq \varphi(x,y,z) \tag{3.3}$$

for all $x,y,z\in V$, then there exists a unique quadratic mapping $F : V \to Y$ such that

(3.4)

$$\|f(x)-F(x)\| \leq \begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(k^i x, k^i x, k^i x)}{k^{2i+2}} & \text{if } \varphi \text{ satisfites (3.1)}, \\ \sum_{i=0}^{\infty} k^{2i}\varphi\big(\frac{x}{k^{i+1}}, \frac{x}{k^{i+1}}, \frac{x}{k^{i+1}}\big) & \text{if } \varphi \text{ satisfites (3.2)} \end{cases}$$

for all $x\in V$.

Proof. We will prove the theorem in two cases, either φ satisfies (3.1) or φ satisfies (3.2).

Case 1. Let φ satisfy (3.1). It follows from (3.3) that

$$\begin{aligned} \left\| \frac{f(k^n x)}{k^{2n}} - \frac{f(k^{n+m}x)}{k^{2n+2m}} \right\| &= \sum_{i=n}^{n+m-1} \left\| \frac{f(k^i x)}{k^{2i}} - \frac{f(k^{i+1}x)}{k^{2i+2}} \right\| \\ &\leq \sum_{i=n}^{n+m-1} \frac{\| -E_{a,b,c}f(k^i x, k^i x, k^i x)\|}{k^{2i+2}} \\ &\leq \sum_{i=n}^{n+m-1} \frac{\varphi(k^i x, k^i x, k^i x)}{k^{2i+2}} \end{aligned} \tag{3.5}$$

for all $x\in V$. So, it is easy to show that the sequence $\{\frac{f(k^n x)}{k^{2n}}\}$ is a Cauchy sequence for all $x\in V$. Since Y is complete and $f(0)=0$, the sequence $\{\frac{f(k^n x)}{k^{2n}}\}$ converges for all $x\in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}}$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (3.5), we obtain the first inequality in (3.4). From the definition of F and (3.3), we get

$$\begin{aligned}\|E_{a,b,c}F(x,y,z)\| &= \lim_{n\to\infty} \left\| \frac{E_{a,b,c}f\big(k^n x, k^n y, k^n z\big)}{k^{2n}} \right\| \\ &\le \lim_{n\to\infty} \frac{\varphi\big(k^n x, k^n y, k^n z\big)}{k^{2n}} = 0\end{aligned}$$

for all $x, y, z \in V$ i.e., $E_{a,b,c}F(x,y,z) = 0$ for all $x, y, z \in V$. By Lemma 3.1, F is a quadratic mapping. To prove the uniqueness, we assume now that there is another quadratic mapping $F' : V \to W$ which satisfies the first inequality in (3.4). Notice that $F'(x) = \frac{F'(k^n x)}{k^{2n}}$ for all $x \in V$. Using (3.1) and (3.4), we obtain

$$\begin{aligned}\lim_{n\to\infty} \left\| \frac{f(k^n x)}{k^{2n}} - F'(x) \right\| &= \lim_{n\to\infty} \left\| \frac{f(k^n x)}{k^{2n}} - \frac{F'(k^n x)}{k^{2n}} \right\| \\ &\le \sum_{i=0}^{\infty} \frac{\varphi(k^{i+n}x, k^{i+n}x, k^{i+n}x)}{k^{2n+2i+2}} \\ &\le \sum_{i=n}^{\infty} \frac{\varphi(k^i x, k^i x, k^i x)}{k^{2i+2}} \\ &= 0\end{aligned}$$

for all $x \in V$, i.e, $F'(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}} = F(x)$ for all $x \in V$.

Case 2. Let φ satisfy (3.2). It follows from (3.3) that

$$\begin{aligned}&\left\| k^{2n} f\left(\frac{x}{k^n}\right) - k^{2n+2m} f\left(\frac{x}{k^{n+m}}\right) \right\| \\ &\qquad = \sum_{i=n}^{n+m-1} \left\| k^{2i} f\left(\frac{x}{k^i}\right) - k^{2i+2} f\left(\frac{x}{k^{i+1}}\right) \right\| \\ &\qquad \le \sum_{i=n}^{n+m-1} k^{2i} \left\| E_{a,b,c} f\left(\frac{x}{k^{i+1}}, \frac{x}{k^{i+1}}, \frac{x}{k^{i+1}}\right) \right\| \\ &\qquad \le \sum_{i=n}^{n+m-1} k^{2i} \varphi\left(\frac{x}{k^{i+1}}, \frac{x}{k^{i+1}}, \frac{x}{k^{i+1}}\right) \qquad (3.6)\end{aligned}$$

for all $x \in V$. So, it is easy to show that the sequence $\{k^{2n} f(\frac{x}{k^n})\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete and $f(0) = 0$, the sequence $\{k^{2n} f(\frac{x}{k^n})\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n\to\infty} k^{2n} f\left(\frac{x}{k^n}\right)$$

for all $x \in V$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (3.6), we obtain the second inequality in (3.4). From the definition of F and (3.3), we get

$$\begin{aligned} \|E_{a,b,c}F(x,y,z)\| &= \lim_{n\to\infty} \left\| k^{2n} E_{a,b,c} f\left(\frac{x}{k^n}, \frac{y}{k^n}, \frac{z}{k^n}\right)\right\| \\ &\leq \lim_{n\to\infty} k^{2n} \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}, \frac{z}{k^n}\right) \\ &=0 \end{aligned}$$

for all $x, y, z \in V$ i.e., $DF(x, y, z) = 0$ for all $x, y, z \in V$. By Lemma 3.1, f is a quadratic mapping. To prove the uniqueness, we assume now that there is another mapping $F' : V \to W$ which satisfies the second inequality in (3.4). Notice that $F'(x) = k^{2n} F'\left(\frac{x}{k^n}\right)$ for all $x \in V$. Using (3.2) and (3.4), we get

$$\begin{aligned} \lim_{n\to\infty} \left\| k^{2n} f\left(\frac{x}{k^n}\right) - F'(x)\right\| &= \lim_{n\to\infty} \left\| k^{2n} f\left(\frac{x}{k^n}\right) - k^{2n} F'\left(\frac{x}{k^n}\right)\right\| \\ &\leq \sum_{i=0}^{\infty} k^{2n+2i} \varphi\left(\frac{x}{k^{n+i}}, \frac{x}{k^{n+i}}, \frac{x}{k^{n+i}}\right) \\ &\leq \sum_{i=n}^{\infty} k^{2i} \varphi\left(\frac{x}{k^i}, \frac{x}{k^i}, \frac{x}{k^i}\right) \\ &= 0 \end{aligned}$$

for all $x \in V$, i.e, $F'(x) = \lim_{n\to\infty} k^{2n} f\left(\frac{x}{k^n}\right) = F(x)$ for all $x \in V$. $\square$

The following corollary follows Theorem 3.2.

Corollary 3.3. *Suppose that a, b, c are given as in Theorem 3.2 with $|a + b + c| \neq 1$, and p, θ are positive real constants with $p \neq 2$. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|E_{a,b,c} f(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{3\theta\|x\|^p}{|k^2 - |k|^p|} \tag{3.7}$$

for all $x \in X$, where $k := a + b + c$.

Proof. First, consider the case $|a + b + c| > 1$. If we put $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$, then φ satisfies (3.1) when $0 < p < 2$ and φ satisfies (3.2) when $p > 2$. Therefore by Theorems 3.2, we obtain the desired inequality (3.7). For the case $|a + b + c| < 1$, we can easily the inequality (3.7) by the similar method. □

Now, we will prove the stability of the functional equation (1.3) for the case $a + b + c = 0$ in the following two theorems.

THEOREM 3.4. *Let a, b and c be nonzero rational numbers with $a + b + c = 0$ and let $\varphi : V^3 \to [0, \infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi((-c)^i x, (-c)^i y, 0)}{c^{2i}} < \infty, \tag{3.8}$$

$$\sum_{i=0}^{\infty} c^{2i} \varphi\left(\frac{x}{(-c)^i}, \frac{y}{(-c)^i}, 0\right) < \infty \tag{3.9}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies the inequality (3.3) for all $x, y, z \in V$ with $f(0) = 0$, then there exists a unique quadratic mapping $F : V \to Y$ such that

(3.10)

$$\|f(x) - F(x)\| \leq \begin{cases} \sum_{i=0}^{\infty} \frac{\varphi((-c)^i x, (-c)^i x, 0)}{c^{2i+2}} & \textit{if } \varphi \textit{ satisfites } (3.8), \\ \sum_{i=0}^{\infty} c^{2i} \varphi\left(\frac{x}{(-c)^{i+1}}, \frac{x}{(-c)^{i+1}}, 0\right) & \textit{if } \varphi \textit{ satisfites } (3.9) \end{cases}$$

for all $x \in V$.

Proof. Since $E_{a,b}f(x, y) = E_{a,b,c}f(x, y, 0)$ for all $x, y \in V$, we get the inequality

$$\|E_{a,b}f(x, y)\| \leq \varphi(x, y, 0) \tag{3.11}$$

for all $x, y \in V$. Hence we obtain the unique quadratic mapping $F : V \to Y$ satisfying the inequality (3.11) from Theorem 2.10 in [6]. □

Similarly the following theorem follows from Theorem 2.4 in [6].

Theorem 3.5. *Let a, b and c be nonzero rational numbers with $a + b + c = 0$ and let $\varphi : V^3 \to [0, \infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x, a^i y, 0)}{a^{2i}} < \infty, \tag{3.12}$$

$$\sum_{i=0}^{\infty} a^{2i} \varphi\Big(\frac{x}{a^i}, \frac{y}{a^i}, 0\Big) < \infty \tag{3.13}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies the inequality (3.3) for all $x, y, z \in V$ with $f(0) = 0$, then there exists a unique quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x, 0, 0)}{a^{2i+2}} & \text{if } \varphi \text{ satisfites (3.12)}, \\ \sum_{i=0}^{\infty} a^{2i} \varphi\big(\frac{x}{a^{i+1}}, 0, 0\big) & \text{if } \varphi \text{ satisfites (3.13)} \end{cases}$$

for all $x \in V$.

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Department of Mathematics Education
Gongju National University of Education
Gongju 32553, Republic of Korea
E-mail: ssjin@gjue.ac.kr

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Department of Mathematics Education
Gongju National University of Education
Gongju 32553, Republic of Korea
E-mail: lyhmzi@pro.gjue.ac.kr